Introduction to Linear Algebra

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SYSTEMS OF LINEAR EQUATIONS

Previously we have seen how to solve a very simple system of equations, namely a linear system with 2 equations and 2 unknowns.

The way these systems are solved are quite straight forward. However, when we have systems of 3,4,... or n equations we need a more systematic way to solve them.

SYSTEMS OF LINEAR EQUATIONS

In this sense is crucial to understand the notation at hand. We will consider a system of **m equations and n unknowns** $x_1, x_2, ..., x_n$:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n & = & b_2 \\ .... & & \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n & = & b_m \end{array} \tag{1}$$

Where $a_{11}, a_{12}, ..., a_{mn}$ are the coefficients of the system, and $b_1, ..., b_m$ are called the right-hand sides.

Note:

1. The equations are linear in the unknowns
2. One or more coefficients might be 0

SYSTEMS OF LINEAR EQUATIONS

A solution to (1) is a list of numbers $s_1, s_2, \ldots, s_n$ that satisfy all the equations simultaneously where $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$.

The system is called

- **Consistent** if it has at least 1 solution
  - **Determined:** if it has only one solution
  - **Undetermined:** if it has infinitely many solutions
- **Inconsistent** if it has no solution

SYSTEMS OF LINEAR EQUATIONS

Example:

T. Haavelmo devised a model of the US economy for the years 1929-1941 based on the following equations:

(i)	$c = 0.712y + 95.05$	(ii)	$x = 0.158(c+x) - 34.3$
(iii)	$y = c + x - s$	(iv)	$x = 93.53$

Here x denotes total investment, y is disposable income, s is the total saving by firms, and c is total consumption. Write the system of equations in the form of (1) when the variables appear in the order x, y, s, and c. Then find the solution of the system.

SYSTEMS OF LINEAR EQUATIONS

Example:

$$\begin{array}{rcrcrcrl} c & - & 0.712y & & & & & = 95.05 \\ 0.158c & & & - & 0.842x & & & = 34.3 \\ c & - & y & + & x & - & s & = 0 \\ & & & & x & & & = 93.53 \end{array}$$

VECTORS

Vectors are just a collection of number disposed in rows (row vector) or columns (column vector). Thus if $\mathbf{a}$ is a $n \times 1$ vector, we write

$$\mathbf{a} = (a_1, a_2, \cdots, a_n)$$

The numbers $a_1, a_2, \cdots, a_n$ are called the **components or coordinates** of the vector, which is referred to as the n-vector.

For example a 3-dimensional vector might be represented in the 3-dimensional space $\mathbb{R}^3$

Let **a** and **b** be two n-dimensional vectors and s and t two scalars, then we can perform the following operations

- summation: $\mathbf{a}+\mathbf{b} = (a_1+b_1, a_2+b_2, \cdots, a_n+b_n)$
- Scalar product: $c = t\mathbf{a} = (ta_1, ta_2, \cdots, ta_n)$

Joining both operations together is said to be a **linear combination**, in symbols

$$t\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + s\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} ta_1+sb_1 \\ ta_2+sb_2 \\ \vdots \\ ta_n+sb_n \end{pmatrix}$$

VECTORS

INNER PRODUCT

Vectors can be multiply between them. However this is a special operation that goes beyond intuition, in this way we define the inner product

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The **inner product** of the n-vectors $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ and $\mathbf{b} = (b_1, b_2, \cdots, b_n)$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

Example: because vectors are just a collection of numbers we can regard $\mathbf{p}$ as a vector with listed prices of some commodities and $\mathbf{x}$ as the list of quantities bought of those commodities. What will we obtain if we perform the inner product?

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Solution: The total amount spent $\mathbf{S} = \mathbf{p} \cdot \mathbf{x} = \sum_{i=1}^{n} p_i x_i$

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Question: What is the maximum that we can spend?

Answer: Total income I, i.e. $\mathbf{p} \cdot \mathbf{x} = \sum_{i=1}^{n} p_i x_i \leq I$

VECTORS
RULES OF INNER PRODUCT

If **a**, **b** and **c** are n-vectors and α is a scalar, then

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $(\alpha \mathbf{a}) \mathbf{b} = \mathbf{a}(\alpha \mathbf{b}) = \alpha(\mathbf{a} \cdot \mathbf{b})$
- $\mathbf{a} \cdot \mathbf{a} > \mathbf{0} \iff \mathbf{a} \neq \mathbf{0}$

MATRICES AND MATRIX OPERATIONS

A **matrix** is a rectangular array of numbers considered as one mathematical object. Matrices are composed of m rows and n columns, the are normally noted in bold letters such as **A**, **B**, etc. In general, they take the form:

$$\mathbf{A} = \left(a_{ij}\right)_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{2}$$

And we say that **A** is of a $m \times n$ **order**

MATRICES AND MATRIX OPERATIONS

A matrix with one column or one row is called a vector.

row vector:

$$\mathbf{v} = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Examples:

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 8 & 5 \\ 7 & 6 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} \sqrt{5} & \frac{5}{4} & -7000 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 2 \\ \pi \\ -\ln\sqrt{3} \\ 0 \end{pmatrix}$$

Where **A** is a 4×2 matrix and $a_{32} = 6$, **r** is a 1×5 row vector and $r_4 = 1$, and **c** is a 4×1 column vector and $c_2 = \pi$

MATRICES AND MATRIX OPERATIONS

Exercise: Construct a 4×3 matrix such that $\mathbf{A} = (a_{ij})_{4\times3}$ with $a_{ij} = 2i - j$

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Solution:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \\ 7 & 6 & 5 \end{pmatrix}$$

MATRICES AND MATRIX OPERATIONS

SUMMATION

Only matrices of the same order can be summed

if $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n}$, then the sum of $\mathbf{A}+\mathbf{B}$ is

$$\mathbf{A}+\mathbf{B} = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij}+b_{ij})_{m\times n}$$

Exercise: If matrices are of different order they are said to be **non-conformable** for summation. What pair of matrices are conformable and non-conformable?

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 8 & 5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 5 & 3 \\ 4 & -7 \\ \sqrt{2} & \pi \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 5 & 3 \\ 4 & -7 \\ 7 & 0 \end{pmatrix}$$

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Solution:

- **A**-**B** and **A**-**C** are non-conformable
- **B**-**C** are conformable

Exercise: Sum the following matrices into $\mathbf{A} + \mathbf{B} = \mathbf{C}$

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 8 & 5 \\ 7 & 6 \\ 1 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 5 & 3 \\ 4 & -7 \\ 7 & 0 \\ \sqrt{2} & \pi \end{pmatrix}$$

Exercise: Sum the following matrices into $\mathbf{A} + \mathbf{B} = \mathbf{C}$

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Solution:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{pmatrix} 4 & 5 \\ 12 & -2 \\ 14 & 6 \\ 1+\sqrt{2} & 1+\pi \end{pmatrix}$$

when a matrix **A** is multiplied by a scalar α, every entry is multiplied by this scalar:

$$\alpha\mathbf{A} = \alpha\left(a_{ij}\right)_{m\times n} = \left(\alpha a_{ij}\right)_{m\times n}$$

Exercise: work out $\alpha\mathbf{A}$ for

$$\alpha = -3$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \\ 7 & 6 & 5 \end{pmatrix}$$

MATRICES AND MATRIX OPERATIONS

SCALAR MULTIPLICATION

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$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \\ 7 & 6 & 5 \end{pmatrix}$$

Solution:

$$\mathbf{A} = \begin{pmatrix} -3 & 0 & 3 \\ -9 & -6 & -3 \\ -15 & -12 & -9 \\ -21 & -18 & -15 \end{pmatrix}$$

MATRICES AND MATRIX OPERATIONS

RULES

Rules for matrix addition and multiplication by a scalar

- $(\mathbf{A}+\mathbf{B})+\mathbf{C}=(\mathbf{A}+\mathbf{B}+\mathbf{C})$
- $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
- $\mathbf{A}+\mathbf{0}=\mathbf{A}$
- $\mathbf{A}-\mathbf{A}=\mathbf{0}$
- $(\alpha+\beta)\mathbf{A}=\alpha\mathbf{A}+\beta\mathbf{A}$
- $\alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}$

Note that because previous definitions of addition and scalar multiplication $3\mathbf{A}=\mathbf{A}+\mathbf{A}+\mathbf{A}$

GAUSSIAN ELIMINATION

Gaussian elimination is a method for solving systems of equations, which uses the method of elimination already seen in a systematic manner.

METHOD:

1. Make a staircase with 1 as the coefficient for each non-zero leading entry
2. produce 0's above each leading entry
3. General solution: expressing the unknowns that occur as leading entries in terms of those unknowns that do not.

GAUSSIAN ELIMINATION

In order to use this method we can:

- Interchange rows
- Multiply any row by an scalar
- Add or subtract any row from any other

GAUSSIAN ELIMINATION

Example: Solve the following system by Gaussian elimination,

$$\begin{aligned} 2x_2 - x_3 &= -7 \\ x_1 + x_2 + 3x_3 &= 2 \\ -3x_1 + 2x_2 + 2x_3 &= -10 \end{aligned}$$

Solution:

1. put the system in matrix form:

$$\left(\begin{array}{ccc|c} 0 & 2 & -1 & -7 \\ 1 & 1 & 3 & 2 \\ -3 & 2 & 2 & -10 \end{array}\right)$$

GAUSSIAN ELIMINATION

2. Interchange 1^{st} and 2^{nd} rows

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 2 & -1 & -7 \\ -3 & 2 & 2 & -10 \end{array}\right)$$

3. Sum to the 3^{rd} row 3 times the 1^{st}

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 2 & -1 & -7 \\ 0 & 5 & 11 & -4 \end{array}\right)$$

GAUSSIAN ELIMINATION

4. Divide the 2^{nd} by 2

$$\left(\begin{array}{ccc:c} 1 & 1 & 3 & 2 \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 5 & 11 & -4 \end{array}\right)$$

5. subtract to the 3^{rd} row 5 times the 2^{nd}

$$\left(\begin{array}{ccc:c} 1 & 1 & 3 & 2 \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & \frac{27}{2} & -\frac{27}{2} \end{array}\right)$$

GAUSSIAN ELIMINATION

6. Multiply the 3^{rd} by $\frac{2}{27}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & 1 \end{array}\right)$$

7. Sum to the 2^{nd} row $\frac{1}{2}$ times the 3^{rd} and subtract from the 1^{st} 3 times the 3^{rd}.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

GAUSSIAN ELIMINATION

8. subtrac the 2$^{\text{nd}}$ from the 1$^{\text{st}}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

9. Sum to the 2$^{\text{nd}}$ row $\frac{1}{2}$ times the 3$^{\text{rd}}$ and subtract from the 1$^{\text{st}}$ 3 times the 3$^{\text{rd}}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

The final solution being the vector $\mathbf{x} = (x_1, x_2, x_3) = (2, -3, 1)$

MATRIX MULTIPLICATION

Matrix multiplication is less intuitive than the previous rules since "normal" operations do not apply.

Suppose that $\mathbf{A} = \left(a_{ij}\right)_{m\times n}$ and that $\mathbf{B} = \left(b_{ij}\right)_{n\times p}$. Then the product $\mathbf{C} = \mathbf{AB}$ is the $m \times p$ matrix $\mathbf{C} = \left(c_{ij}\right)_{m\times p}$, whose element in the ith row and jth column is the **dot (inner) product**:

$$c_{ij} = \mathbf{a_i b_j} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ik} b_{kj} + \cdots + a_{in} b_{nj}$$

of the ith row of **A** and the jth column of **B**.

MATRIX MULTIPLICATION

More graphically:

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ \boxed{a_{i1}} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & \boxed{b_{1j}} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & \boxed{c_{ij}} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

MATRIX MULTIPLICATION

Exercise: for matrices to be **conformable** under multiplication the number of columns of **A** needs to be the same as the number of rows of **B**. Are these matrices conformable? operate if they are.

$$\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 7 & 4 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 8 \\ 3 & 9 \\ 6 & 2 \end{pmatrix}$$

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Solution:

- **AB** is not conformable
- **BA** is conformable

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Solution:

- **AB** is not conformable
- **BA** is conformable

$$\mathbf{C} = \mathbf{BA} = \begin{pmatrix} 56 & 35 \\ 63 & 45 \\ 14 & 30 \end{pmatrix}$$

MATRIX MULTIPLICATION

Exercise: a very special property of matrices is that the **commutative property does not hold**. Multiply the following matrices in both ways, i.e. **AB** and **BA**:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & -1 \\ 5 & 2 \end{pmatrix}$$

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Solution:

$$\mathbf{AB} = \begin{pmatrix} 5 & 2 \\ 17 & 4 \end{pmatrix} \quad \mathbf{BA} = \begin{pmatrix} -2 & -2 \\ 4 & 11 \end{pmatrix}$$

Thanks to this newly introduced notation we can write complex systems of equations in matrix form. Consider the system of equations given in (1):

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\
&\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m
\end{aligned}
$$

We can write the previous system of equations as

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\mathbf{b}} \tag{3}$$

Hence as

$$\mathbf{Ax} = \mathbf{b}$$

Which is much more compact.

Exercise: Write the following system of equations in matrix form

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ x_1 & - & x_2 & + & x_3 & = & 5 \\ 2x_1 & + & 3x_2 & - & x_3 & = & 1 \end{array}$$

MATRIX MULTIPLICATION

SYSTEMS OF EQUATIONS IN MATRIX FORM

Exercise: Write the following system of equations in matrix form

$$\begin{array}{ccccccc} x_1 & + & 2x_2 & + & 3x_3 & = & 4 \\ x_1 & - & x_2 & + & x_3 & = & 5 \\ 2x_1 & + & 3x_2 & - & x_3 & = & 1 \end{array}$$

Solution:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$$

Rules of Matrix Multiplication

- Associative: $(\mathbf{AB})\,\mathbf{C} = \mathbf{A}\,(\mathbf{BC})$
- Left distributive law: $\mathbf{A}\,(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
- Right distributive law: $(\mathbf{A}+\mathbf{B})\,\mathbf{C} = \mathbf{AC}+\mathbf{BC}$
- Associative Scalar multiplication:$(\alpha\mathbf{A})\,\mathbf{B} = \mathbf{A}\,(\alpha\mathbf{B}) = \alpha\,(\mathbf{AB})$
- **NO** commutative: $\mathbf{AB} \neq \mathbf{BA}$

MATRIX MULTIPLICATION

POWER MATRIX

The most important family of matrices are those that are **square**, in other words the ones that have the same number of rows and columns $m = n$.

MATRIX MULTIPLICATION

POWER MATRIX

The most important family of matrices are those that are **square**, in other words the ones that have the same number of rows and columns $m = n$.

Power of Matrices: if $\mathbf{A}$ is a square matrix, the associative law allows us to write $\mathbf{AA}$ as $\mathbf{A}^2$. In general,

$$\mathbf{A}^n = \mathbf{AA}\cdots\mathbf{A} \quad \text{(n times)}$$

MATRIX MULTIPLICATION

POWER MATRIX

Exercise: Let $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Compute A^2, A^3, A^4. Then guess the general rule A^n and prove it by induction.

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Solution:

$$\mathbf{A}^2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}^3 = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}^4 = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$$

MATRIX MULTIPLICATION

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Solution:

$$\mathbf{A}^2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}^3 = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}^4 = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{A}^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$$

Induction step

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -k-1 \\ 0 & 1 \end{pmatrix}$$

Since this is also true for $n = 1$ it completes the proof.

MATRIX MULTIPLICATION

IDENTITY MATRIX

The **Identity Matrix** is the matrix of order n, denoted by $\mathbf{I}_n$ or $\mathbf{I}$ having ones along the main diagonal and zeros elsewhere:

$$\mathbf{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Which has the property that

$$\mathbf{I}_n\mathbf{A} = \mathbf{A}\mathbf{I}_n = \mathbf{A}$$

And that corresponds to 1 in the real number system

THE TRANSPOSE

The transpose of a matrix $\mathbf{A}'$ is a matrix whose rows and columns have been swapped. Particularly

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \Rightarrow \mathbf{A}' = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}$$

So $A' = \left(a'_{ij}\right)$ where $a'_{ij} = a_{ji}$

THE TRANSPOSE

SYMMETRIC MATRIX

Another important kind of matrix are the square ones that are **symmetric** along the main diagonal. They have the property that are equal to its own transpose:

$$\text{The matrix } \mathbf{A} \text{ is sysmmetric} \iff \mathbf{A} = \mathbf{A}'$$

Example:

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & -1 & 8 \\ 5 & 8 & -1 \end{pmatrix}$$

THE TRANSPOSE

RULES

Rules of transposition:

- $\left(A'\right)' = A$
- $(\mathbf{A}+\mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
- $(\alpha\mathbf{A})' = \alpha\mathbf{A}'$
- $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$

THE TRANSPOSE

RULES

Exercise: For what values of a is the matrix A symmetric? **Example:**

$$\begin{pmatrix} a & a^2-1 & -3 \\ a+1 & 2 & a^2+4 \\ -3 & 4a & -1 \end{pmatrix}$$

Exercise: For what values of a is the matrix A symmetric? **Example:**

$$\begin{pmatrix} a & a^2-1 & -3 \\ a+1 & 2 & a^2+4 \\ -3 & 4a & -1 \end{pmatrix}$$

Solution: There are two conditions

1. $a+1 = a^2-1 \iff a+1 = (a+1)(a-1)$ and there are two cases:
   - $a = 1 \Rightarrow 2 = 0$ and there is contradiction
   - $a \neq 1 \Rightarrow a = 2$ and this one is a candidate
2. $4a = a^2+4 \iff a^2-4a+4 = 0 \iff (a-2)^2 = 0 \Rightarrow a = 2$

The only one option that coincides in both conditions is $a = 2$, which is the solution.

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DETERMINANTS

Determinants are important in many are areas that are of interest for economist.

Basically, the determinant is the **area expansion** of the transformation induced by a matrix, or in other words is the incremental factor that a volume experiences when multiplied by a matrix.

The main feature of the determinant is that it will allow us to rapidly know if a system of equations has solution or not.

Example of the area expansion caused by a matrix transformation
Consider the vector and the matrix

$$\mathbf{v} = (1, 1)$$

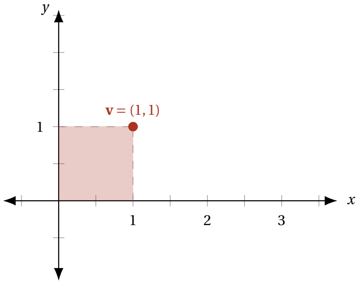
$$\mathbf{T} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

And apply the matrix transformation of **T** to **v** in order to have the new vector v'

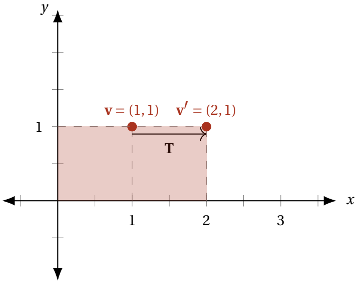
$$T \cdot \mathbf{v} = \mathbf{v}' \iff \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \cdot (1, 1) = (2, 1)$$

Because the volume (area in this case) between the origin and the vector has doubled the determinant will be 2.

Graphically



Area of vector v



Area under the transformation

2 ORDER DETERMINANTS

Consider the following system of equations

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{array}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{4}$$

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$$\begin{matrix} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{matrix}, \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{4}$$

Solving the system yields (make sure you do it!!!)

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}, \quad x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}}$$

2 ORDER DETERMINANTS

Both solutions have common denominator $\mathbf{D} = a_{11}a_{22} - a_{21}a_{12}$.

For the system (4) to have a solution $D \neq 0$, in this sense D determines the solvability of the system, hence the **determinant**.

The determinant of $\mathbf{A}$ is denoted by $det(\mathbf{A})$ or $|\mathbf{A}|$, thus

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \tag{5}$$

2 ORDER DETERMINANTS

CALCULATION

Consider the $det(\mathbf{A})$ in (5).

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

The following steps are valid to work out determinants of a square matrix of order 2:

1. Multiply the elements of the main diagonal
2. Multiply the elements of the off-diagonal elements
3. subtract the second step from the first

3 ORDER DETERMINANTS

Consider the system of three linear equations in three unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

The solutions to this system (x_1, x_2, x_3) will have a **common denominator**, namely the determinant.

Computation

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{cases} a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} \\ -a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{31} \end{cases}$$

3 ORDER DETERMINANTS

Exercise: Work out the determinant of the following matrix

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Solution:

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & -1 \end{vmatrix} = -3 + 0 + 6 - 0 + 4 - 4 = 3$$

DETERMINANTS RULES

The generalisation of determinant for $n \times n$ matrices is not straight forward and we will not see it. However, the next rules apply for the matrix A_n

- If all the elements in a row or column of $\mathbf{A}$ are $\mathbf{0}$, then $|\mathbf{A}| = 0$

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- If α is a real number, $|\alpha\mathbf{A}| = \alpha^n|\mathbf{A}|$

INVERSE OF A MATRIX

In the real number system, for every non-zero number a there is another number such that $a \cdot a^{-1} = 1$. We call this number a^{-1} the inverse of a.

The simile to 1 in the matrix system is the identity matrix I_n, in this sense we say that **X** is the inverse of **A** if the next relation applies

$$\mathbf{AX} = \mathbf{XA} = \mathbf{I} \tag{6}$$

Then **A** is said to be invertible.

INVERSE OF A MATRIX

The sufficient and necessary condition for a matrix to be **invertible** is:

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The inverse of a matrix is unique, indeed assuming that **Y** and **X** are inverses of **A** and by (6)

$$\mathbf{Y} = \mathbf{IY} = (\mathbf{XA})\,\mathbf{Y} = \mathbf{X}\,(\mathbf{AY}) = \mathbf{XI} = \mathbf{X}$$

provided it exists

INVERSE OF A MATRIX

INVERSE PROPERTIES

Let $\mathbf{A}$ and $\mathbf{B}$ be invertible $n \times n$ matrices. Then:

- $\mathbf{A}^{-1}$ is invertible, and $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$
- $\mathbf{AB}$ is invertible, and $(\mathbf{AB})^{-1} = (\mathbf{B})^{-1}(\mathbf{A})^{-1}$
- The transpose $\mathbf{A}'$ is invertible, and $\left(\mathbf{A}'\right)^{-1} = \left(\mathbf{A}^{-1}\right)'$
- $(c\mathbf{A})^{-1} = c^{-1}\mathbf{A}^{-1}$ whenever c is a number $\neq 0$

INVERSE OF A MATRIX

FINDING THE INVERSE

Suppose we have a matrix $\mathbf{A}$, then we need to find a matrix $\mathbf{B}$ such that $\mathbf{A} \cdot \mathbf{B} = I$. An efficient way to look for $\mathbf{B}$, provided it exists, is

1. From the $n \times 2n$ matrix $(\mathbf{A} : \mathbf{I})$
2. Apply elementary operations to transform it into an $n \times 2n$ matrix $(\mathbf{I} : \mathbf{B})$
3. It follows $\mathbf{B} = \mathbf{A}^{-1}$

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Example: Find the inverse of

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

Solution:

1.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \end{array}\right)$$

2. Interchange the 2^{nd} and 3^{rd} rows

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \\ 1 & 3 & 4 & 0 & 1 & 0 \end{array}\right)$$

INVERSE OF A MATRIX

FINDING THE INVERSE

3. Subtract the 1st row from 2nd and 3rd rows

$$\left(\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

4. Subtract 3 times the 2nd and 3rd rows from the 1st row

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

INVERSE OF A MATRIX

FINDING THE INVERSE

3. Subtract the 1^{st} row from 2^{nd} and 3^{rd} rows

$$\left(\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

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INVERSE OF A MATRIX

IMPORTANCE

The most important feature of the inverse is that it will allow us to solve complex systems of equations such as (1)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\dots\dots\dots\dots\dots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}
$$

INVERSE OF A MATRIX

IMPORTANCE

As before you can write this system in matrix form like in (3)

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\mathbf{b}}$$

Hence as

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Taking the inverse of **A** we would have solved the system in just one step:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$